A HAUSDORFF–YOUNG INEQUALITY FOR LOCALLY COMPACT QUANTUM GROUPS

TOM COONEY

Abstract. Let G be a locally compact abelian group with dual group $\hat{G}$. The Hausdorff–Young theorem states that if $f \in L^p(G)$, where $1 \leq p \leq 2$, then its Fourier transform $\mathcal{F}_p(f)$ belongs to $L^q(\hat{G})$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and $\|\mathcal{F}_p(f)\|_q \leq \|f\|_p$. Kunze and Terp extended this to unimodular and locally compact groups, respectively. We further generalize this result to an arbitrary locally compact quantum group $\mathbb{G}$ by defining a Fourier transform $\mathcal{F}_p : L_p(\mathbb{G}) \to L_q(\hat{\mathbb{G}})$ and showing that this Fourier transform satisfies the Hausdorff–Young inequality.

1. Introduction

Let G be a locally compact abelian group with Haar measure μ and dual group $\hat{G}$. The Fourier transform of a function $f \in L_1(G, \mu)$ is defined by

$$\mathcal{F}_1(f)(\xi) = \hat{f}(\xi) = \int_G f(s)\overline{\xi(s)}d\mu(s), \qquad \xi \in \hat{G}.$$

Clearly, $\|\mathcal{F}_1(f)\|_\infty \leq \|f\|_1$. For a suitably normalized Haar measure $\hat{\mu}$ on $\hat{G}$, if $f \in L_1(G, \mu) \cap L_2(G, \mu)$, then we have $\|\mathcal{F}_1(f)\|_2 = \|f\|_2$ and the Fourier transform extends to a unitary, $\mathcal{F}_2$, from $L_2(G, \mu)$ onto $L_2(\hat{G}, \hat{\mu})$. Interpolating between these cases yields the Hausdorff–Young inequality: for $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{F}_p : L_p(G, \mu) \to L_q(\hat{G}, \hat{\mu})$ is a contraction, i.e., $\|\mathcal{F}_p(f)\|_q \leq \|f\|_p$ (see, for example, [3]).

For $f \in L_1(G, \mu)$, let $\lambda(f)$ denote the operator on $L_2(G, \mu)$ given by $\lambda(f)(g) = f * g$. As $\widehat{f * g} = \hat{f}\hat{g}$, the operator $\lambda(f)$ is unitarily equivalent to the operator $\hat{f} \in L_\infty(\hat{G}, \hat{\mu})$ acting by multiplication on $L_2(\hat{G}, \hat{\mu})$. Inspired by this, Kunze dealt with the unimodular group case by making the definition $\mathcal{F}_1(f) = \lambda(f)$. The group von Neumann algebra $L(G)$ is generated by $\{\lambda(f) \mid f \in L_1(G)\}$ and plays the role of $L_\infty(\hat{G})$. Using the noncommutative L_p spaces associated with a trace on $L(G)$, Kunze showed in [10] that the Hausdorff–Young inequality holds for unimodular groups.

Terp extended the Hausdorff–Young inequality to locally compact groups in [15] by using the following definition for the Fourier transform:

Definition 1.1. Let G be a locally compact group with modular function Δ. Let $f \in L_p(G)$, $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. The L_p-*Fourier transform of* f is the operator $\mathcal{F}_p(f)$ on $L_2(G)$ given by

$$\mathcal{F}_p(f)\xi = f * \Delta^{1/q}\xi, \quad \xi \in \mathcal{D}(\mathcal{F}_p(f)),$$

where $\mathcal{D}(\mathcal{F}_p(f)) = \{\xi \in L_2(G) \mid f * \Delta^{1/q}\xi \in L_2(G)\}$.

Terp showed that $\mathcal{F}_p(f) \in L_q(L(G))$, where $L_q(L(G))$ is a non-commutative L_q space constructed using the Plancherel weight on the group von Neumann algebra $L(G)$, and showed that the Hausdorff–Young inequality holds. This is the definition that shall be extended to the locally compact quantum group case.

For the convenience of the reader, the main result is now presented. Let $\mathbb{G}$ be a locally compact quantum group with dual locally compact quantum group $\hat{\mathbb{G}}$. Let φ be the normal, semi-finite, faithful (nsf) Haar weight on $L_\infty(\mathbb{G})$ and let φ' be a nsf weight on the commutant $L_\infty(\mathbb{G})'$. Let $d = \frac{d\varphi}{d\varphi'}$ be the spatial derivative of φ with respect to φ'. Similarly, let $\hat{\varphi}$ be the Haar weight on $L_\infty(\hat{\mathbb{G}})$, let $\hat{\varphi}'$ be a nsf weight on $L_\infty(\hat{\mathbb{G}})'$, and let $\hat{d} = \frac{d\hat{\varphi}}{d\hat{\varphi}'}$. Let $L_p(\mathbb{G})$ and $L_p(\hat{\mathbb{G}})$ be the spatial non-commutative L_p-spaces constructed using φ' and $\hat{\varphi}'$ respectively. See Section 2.3 for further details.

University of Illinois at Urbana-Champaign, Email:tcooney@math.uiuc.edu.

Let $\pi_l(\mathfrak{A}_0)$ be the set of entire elements x such that $\sigma_\alpha^\varphi(x) \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ for all $\alpha \in \mathbb{C}$, and let $\pi_l(\mathfrak{A}_0^2) = \{xy : x, y \in \pi_l(\mathfrak{A}_0)\}$. Let L be the set of $x \in L_\infty(\mathbb{G})$ such that there is a normal linear functional φ_x in $L_1(\mathbb{G})$, the predual of $L_\infty(\mathbb{G})$, satisfying $\varphi_x(y) = \varphi(yx)$ for $y \in \pi_l(\mathfrak{A}_0^2)$. For each $x \in L$, there is a corresponding element $U_p(j^*(x))$ in $L_p(\mathbb{G})$. See Section 2.4 for further details.

Theorem 3.5. *The L_p-Fourier transform $\mathcal{F}_p$ is the map from $L_p(\mathbb{G})$ to $L_q(\hat{\mathbb{G}})$ such that*

$$\mathcal{F}_p\left(U_p(j^*(x)\right) = \lambda\left(\varphi_x\right)\hat{d}^{1/q},$$

for $x \in L$. This map is a contraction, i.e., $\|\mathcal{F}_p\| \leq 1$.

For $x \in \pi_l(\mathfrak{A}_0^2)$, there is a more explicit description of $\mathcal{F}_p$:

$$\mathcal{F}_p\left(d^{1/2p}\sigma_{i/2p}^\varphi(x)d^{1/2p}\right) = \lambda(\varphi_x)\hat{d}^{1/q}, \quad x \in \pi_l(\mathfrak{A}_0^2).$$

If φ is a state, we have the simpler expression:

$$\mathcal{F}_p(xd^{1/p}) = \lambda(\varphi_x)\hat{d}^{1/q}, \qquad x \in L_\infty(\mathbb{G}).$$

The cases $p = 1$ and $p = 2$ of the Fourier transform are implicit in the construction of the dual locally compact quantum group, as found in [11], but this does not consider the Fourier transform as a map between non-commutative L_p-spaces. In this work, we identify the operators in $L_p(\mathbb{G})$ and $L_q(\hat{\mathbb{G}})$ that lead to a suitable definition for an L_p-Fourier transform. As in the classical case, the Hausdorff–Young inequality will then be proved by interpolating between the cases $p = 1$ and $p = 2$. We consider a way of viewing non-commutative L_p-spaces as interpolation spaces that is due to Izumi, [6]. We identify the intersection of $L_1(\mathbb{G})$ and $L_2(\mathbb{G})$ under the embedding Izumi's method provides. We approximate elements in this intersection by elements in $\pi_l(\mathfrak{A}_0^2)$. We then show that the Fourier transforms $\mathcal{F}_1 : L_1(\mathbb{G}) \to L_\infty(\hat{\mathbb{G}})$ and $\mathcal{F}_2 : L_2(\mathbb{G}) \to L_2(\hat{\mathbb{G}})$ agree on $L_1(\mathbb{G}) \cap L_2(\mathbb{G})$. Interpolation now yields the Hausdorff–Young inequality for locally compact quantum groups.

We note that the non-commutative L_p spaces are (up to isometric isomorphism) independent of the choice of the weights on the commutants and thus so are the results in this paper. However the connection with the group case is clearer when we work with the following choice of weights on the commutants, $\varphi' = \varphi(J \cdot J)$ and $\hat{\varphi}' = \hat{\varphi}(\hat{J} \cdot \hat{J})$. Suppose G is a locally compact group with modular function Δ. This defines an unbounded operator acting by multiplication on $L_2(G)$, $(\Delta f)(s) = \Delta(s)f(s)$, for $s \in G$. The left Haar weight φ is given by integration against the left Haar measure. If $\varphi' = \varphi(J \cdot J)$, then $d = \frac{d\varphi}{d\varphi'} = 1$. Let $\hat{\varphi}$ be the Plancherel weight on the group von Neumann algebra $L(G)$ (for example, see Section VII.3 of [13]). If we let $\hat{\varphi}' = \hat{\varphi}(\hat{J} \cdot \hat{J})$, then $\hat{d} = \frac{d\hat{\varphi}}{d\hat{\varphi}'} = \Delta$ (see [14] or [15]). Terp's Fourier transform can then be written as $\mathcal{F}_p(f) = \lambda(f)\hat{d}^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. This shows the locally compact quantum group definition generalizes the locally compact group case. It is to preserve this connection with the group case that Hilsum's construction of non-commutative L_p-spaces will be used. The L_p-Fourier transform is also compatible with the Fourier transform from a suitable subset of $L_\infty(\mathbb{G})$ into $L_\infty(\hat{\mathbb{G}})$ considered in [8] and [17].

In Section 2, we recall results from the modular theory of von Neumann algebras, the construction of the dual locally compact group, the construction of non-commutative L_p spaces, and how non-commutative L_p spaces can be considered as interpolation spaces. In Section 3, we define the Fourier transform and prove the Hausdorff–Young inequality for locally compact quantum groups.

2. Background

2.1. **Hilbert Algebras.** Let M be a von Neumann algebra and φ a normal, semi-finite, faithful weight on M. Let

$$\mathfrak{N}_\varphi = \{x \in M : \varphi(x^*x) < \infty\} \text{ and } \mathfrak{M}_\varphi = \operatorname{span}\{y^*x : y, x \in \mathfrak{N}_\varphi\}.$$

The Hilbert space $\mathcal{H} = \mathcal{H}_\varphi$ is the completion of $\mathfrak{N}_\varphi$ with respect to the inner product $(x \mid y) = \varphi(y^*x)$ and $\Lambda = \Lambda_\varphi$ is the inclusion map $\mathfrak{N}_\varphi \to \mathcal{H}$. Let $\mathfrak{A} = \mathfrak{A}_\varphi = \Lambda(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*)$. This is a left Hilbert algebra with multiplication $\Lambda(x)\Lambda(y) = \Lambda(xy)$ and adjoint $\Lambda(x)^\# = \Lambda(x^*)$. If $x \in \mathfrak{A}$, then $\Lambda(y) \mapsto \Lambda(xy)$ extends to a bounded operator $\pi_l(\Lambda(x))$ on $\mathcal{H}$. The von Neumann algebra M can and will be identified with $\pi_l(\mathfrak{A})'' \subset B(\mathcal{H})$ and then $\pi_l(\Lambda(x)) = x$.

Let $S = S_\varphi$ be the closure of the map $\Lambda(x) \mapsto \Lambda(x^*)$, with polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$. We denote the modular conjugation, modular operator, and modular automorphism group by $J = J_\varphi$, $\Delta = \Delta_\varphi$, and $\sigma_t^\varphi(\,\cdot\,) = \Delta^{it} \cdot \Delta^{-it}$, for $t \in \mathbb{R}$, respectively.

In the usual way, construct the right Hilbert algebra $\mathfrak{A}'_\varphi = \mathfrak{A}'$ and the Tomita algebra $\mathfrak{A}_{0,\varphi} = \mathfrak{A}_0$,

$$\mathfrak{A}_0 = \left\{ \xi \in \bigcap_{n\in\mathbb{Z}} \mathcal{D}(\Delta^n) \,:\, \Delta^n \xi \in \mathfrak{A}, n \in \mathbb{Z} \right\} \subset \Lambda(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*).$$

An element $x \in M$ is entire if the map $t \mapsto \sigma_t(x)$ extends to an analytic function $\mathbb{C} \to M$, $\alpha \mapsto \sigma_\alpha^\varphi(x)$. The set of entire elements includes $\pi_l(\mathfrak{A}_0)$, and if $x \in \pi_l(\mathfrak{A}_0)$, then $\sigma_\alpha^\varphi(x) \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, for all $\alpha \in \mathbb{C}$. We will denote $\operatorname{span}\{\Lambda(yx) \,:\, y, x \in \pi_l(\mathfrak{A}_0)\}$ by $\mathfrak{A}_0^2$.

The following two results will be used later.

Theorem 2.1. *Suppose that $x \in \mathfrak{N}_\varphi$. There exists a sequence $(\Lambda(x_n)) \subset \mathfrak{A}_0^2$ such that*

(1) $\lim_{n\to\infty} \|\Lambda(x_n) - \Lambda(x)\| = 0$,
(2) *x_n converges to x in the σ-strong operator topology, and*
(3) $\|x_n\| \le \|x\|$.

Proof. We apply an approximation result due to Haagerup to the Tomita algebra. See [4] or Theorem VI.1.26 in [13]. These sources state that the sequence lies in $\mathfrak{A}_0$ but, examining the proof, we see that the stronger statement that $\Lambda(x_n) \in \mathfrak{A}_0^2$ holds. □

The following standard result can be found in Section 2.18 in [12].

Proposition 2.2. *For all $\alpha \in \mathbb{C}$, $a \in \pi_l(\mathfrak{A}_0)$, and $z \in \mathfrak{M}_\varphi$,*

$$\varphi(z\sigma_\alpha^\varphi(a)) = \varphi(\sigma_{\alpha+i}^\varphi(a)z).$$

For further details concerning Hilbert algebras and Tomita algebras, see [12] and [13].

2.2. **Locally Compact Quantum Groups.** A *locally compact quantum group* $\mathbb{G} = (M, \Gamma, \varphi, \varphi_r)$ consists of

- a von Neumann algebra M,
- a normal, unital, $*$-homomorphism Γ from $M \to M \overline{\otimes} M$ such that $(\Gamma \otimes \iota) \circ \Gamma = (\iota \otimes \Gamma) \circ \Gamma$,
- a nsf weight φ on M such that for $\omega \in M_*^+$, $x \in \mathfrak{M}_\varphi^+$, $\varphi((\omega \otimes \iota)\Gamma(x)) = \varphi(x)\omega(1)$,
- a nsf weight φ_r on M such that for $\omega \in M_*^+$, $x \in \mathfrak{M}_{\varphi_r}^+$, $\varphi_r((\iota \otimes \omega)\Gamma(x)) = \varphi_r(x)\omega(1)$.

Here $\overline{\otimes}$ denotes the von Neumann algebra tensor product. For further details on Kustermans and Vaes's definition of locally compact quantum groups and their construction of the dual locally compact quantum group, see [11].

If G is a locally compact group, then the corresponding von Neumann algebra M is $L_\infty(G, \mu)$, the comultiplication is the map $\Gamma : L_\infty(G) \to L_\infty(G \times G)$ given by $\Gamma(f)(s,t) = f(st)$, and the co-associativity condition corresponds to $f((st)u) = f(s(tu))$. The weights correspond to integrating against the left and right Haar measures on G.

The von Neumann algebra M will often be denoted by $L_\infty(\mathbb{G})$, its predual M_* by $L_1(\mathbb{G})$, and will be taken to be acting standardly on $\mathcal{H} = \mathcal{H}_\varphi$. The *multiplicative unitary* W of $\mathbb{G}$ is the unitary W on $\mathcal{H} \otimes \mathcal{H}$ (in fact, $W \in M \overline{\otimes} \hat{M}$) determined by

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Gamma(y)(x \otimes 1)),$$

for all $x, y \in \mathfrak{N}_\varphi$. This unitary implements the comultiplication: for $x \in L_\infty(\mathbb{G})$,

$$\Gamma(x) = W(1 \otimes x)W^*.$$

The dual locally compact group will be denoted by $\hat{\mathbb{G}} = (L_\infty(\hat{\mathbb{G}}), \hat{\Gamma}, \hat{\varphi}, \hat{\psi})$. A symbol marked with $\hat{}$ denotes an object with the same properties as the unmarked symbol but now defined in the context of the dual locally compact quantum group. We recall how to construct $L_\infty(\hat{\mathbb{G}})$ and $\hat{\varphi}$.

If $\omega_1, \omega_2 \in L_1(\mathbb{G})$, $(\omega_1 * \omega_2)(x) = (\omega_1 \otimes \omega_2)(\Gamma(x))$ defines a multiplication on the predual $L_1(\mathbb{G})$ (which is convolution in the group case). Define the Fourier representation λ by $\lambda(\omega) = (\omega \otimes 1)W$ for $\omega \in L_1(\mathbb{G})$; this satisfies $\lambda(\omega_1 * \omega_2) = \lambda(\omega_1)\lambda(\omega_2)$. The dual locally compact quantum group $\hat{\mathbb{G}}$ has as its von Neumann algebra

$L_\infty(\hat{\mathbb{G}}) = \lambda(L_1(\mathbb{G}))''$, with the comultiplication $\hat{\Gamma}$ determined by the multiplicative unitary $\hat{W} = \Sigma W^* \Sigma$, where $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$.

To construct a left Haar weight $\hat{\varphi}$, consider the set

$$\mathcal{I} = \{\omega \in L_1(\mathbb{G}) \mid \exists C \in \mathbb{R}^+ : |\omega(x^*)| \leq M\|\Lambda(x)\| \text{ for all } x \in \mathfrak{N}_\varphi\}.$$

By the Riesz Representation Theorem, there then exists $\xi(\omega) \in \mathcal{H}$ such that

$$\omega(x^*) = (\xi(\omega) \mid \Lambda(x)), \qquad x \in \mathfrak{N}_\varphi.$$

The set $\lambda(\mathcal{I})$ is a σ-strong*-norm core for the the unique σ-strong*-norm closed linear map $\hat{\Lambda}$ such that $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega)$. The dual weight $\hat{\varphi}$ is the unique normal, semi-finite, faithful weight on $L_\infty(\hat{\mathbb{G}})$ having the triple $(\mathcal{H}, \iota, \hat{\Lambda})$ as its GNS construction. Here ι is the action $\iota(\lambda(\omega_1))\xi(\omega_2) = \xi(\omega_1 * \omega_2)$. Note that $\omega \in \mathcal{I}$ implies that $\lambda(\omega) \in \mathfrak{N}_{\hat{\varphi}}$. This identifies the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{H}_{\hat{\varphi}}$ and so the $p = 2$ case of the Hausdorff–Young inequality is implicit in this construction.

2.3. **Non-commutative L_p spaces.** Non-commutative L_p spaces in the weight case were introduced by Haagerup. Here an alternate approach due to Connes [2] and Hilsum [5] is used. These spatial non-commutative L_p spaces are isometrically isomorphic to those introduced by Haagerup (see [14] for further details).

Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let ψ be a nsf weight on its commutant M'. Let $\mathfrak{N}_\psi = \{x \in M' : \psi(x^*x) < \infty\}$, let $\mathcal{H}_\psi$ be the Hilbert space completion of $\mathfrak{N}_\psi$, and Λ_ψ the inclusion of $\mathfrak{N}_\psi$ into $\mathcal{H}_\psi$. The set of ψ-bounded vectors is

$$\mathcal{D}(\mathcal{H}, \psi) = \{\xi \in \mathcal{H} : \exists C > 0, \|y\xi\|_{\mathcal{H}} \leq C\|\Lambda_\psi(y)\|_{\mathcal{H}_\psi}\},$$

and for all $\xi \in \mathcal{D}(\mathcal{H}, \psi)$, $\Lambda_\psi(y) \mapsto y\xi$ extends to a bounded operator $R^\psi(\xi)$ from $\mathcal{H}_\psi$ to $\mathcal{H}$. If φ is a normal semi-finite weight on M, then its spatial derivative $\frac{d\varphi}{d\psi}$ is the largest positive self-adjoint operator on $\mathcal{H}$ such that

$$(\frac{d\varphi}{d\psi}\xi \mid \xi) = \varphi(R^\psi(\xi)R^\psi(\xi)^*), \qquad \forall \xi \in \mathcal{D}(\mathcal{H}, \psi).$$

The spatial non-commutative L_p spaces are defined as follows: $L_1(M, \psi)$ is the set of closed, densely defined operators with polar decomposition $x = u|x|$ such that $u \in M$ and there exists $\varphi \in M_{*,+}$ with $\frac{d\varphi}{d\psi} = |x|$. Setting $\|x\|_1 = \varphi(1) = \|\varphi\|_{M_*}$ yields an isometric isomorphism between $L_1(M, \psi)$ and M_*. Similarly,

$$L_p(M, \psi) = \{x = u|x| : u \in M, \exists \varphi \in M_* \text{ such that } |x|^p = \frac{d\varphi}{d\psi}\}, \qquad \|x\|_p = \varphi(1)^{1/p}.$$

These spaces are isometrically isomorphic to the L_p spaces of Haagerup and are thus independent of the choice of the weight ψ.

Using results of Terp's, we can write down dense subspaces of $L_p(M, \psi)$. The following can be found as Theorems 22, 26, and 27 from [16].

Theorem 2.3. *Fix a normal semi-finite faithful weight φ_0 on M and let $d = \frac{d\varphi_0}{d\psi}$.*

Let $x \in \mathfrak{N}_\varphi$ and $2 \leq p < \infty$. Then $xd^{1/p}$ is preclosed, its closure (also denoted by $xd^{1/p}$) is in $L_p(M, \psi)$ and the set of operators $\{xd^{1/p} : x \in \mathfrak{N}_\varphi\}$ is dense in $L_p(M, \psi)$. Furthermore,

$$\|xd^{1/2}\|_2 = \|\Lambda(x)\|$$

and the map $x \mapsto xd^{1/2} : \mathfrak{N}_\varphi \to L_2(M, \psi)$ extends to a linear isometry of $\mathcal{H}$ onto $L_2(M, \psi)$.

Similarly, if $1 \leq p < \infty$, then $\{d^{1/2p}xd^{1/2p} : x \in \mathfrak{M}_\varphi\}$ is dense in $L_p(M, \psi)$.

Theorem 2.4. *Let $x \in \pi_l(\mathfrak{A}_0)$. Then for $\alpha \in [0, \frac{1}{2}]$,*

$$xd^\alpha = d^\alpha \sigma^\varphi_{i\alpha}(x).$$

Proof. This has been proven in the context of Haagerup's construction of non-commutative L^p-spaces in [7]. It can be directly proved in the context of Hilsum's construction of non-commutative L^p-spaces by combining Lemma 2.6 from [15] with Lemma 22, Theorem 22, and Theorem 26 from [16]. □

2.4. **Izumi's Interpolation Method.** Let M be a von Neumann algebra with normal, faithful, semi-finite weight φ acting faithfully on $\mathcal{H} = \mathcal{H}_\varphi$. Let $L_p(M, \varphi)$ be the non-commutative L_p-space constructed according to Hilsum's method, using a normal semi-finite weight on the commutant M' (for example, $\varphi' = \varphi(J \cdot J)$). Kosaki showed in [9] that if φ is a state, then the spaces $L_p(M, \varphi)$ can be viewed as interpolation spaces with $M \subset L_p(M, \varphi) \subset M_*$. For each $\eta \in [0, 1]$, his approach provides an embedding such that for $x \in M \subset L_p(M)$, the corresponding element of $L_p(M)$ is $d^{\eta/2p} x d^{(1-\eta)/2p}$.

Terp extended this to the case where φ is a weight, but only for the symmeteric embedding ($\eta = \frac{1}{2}$). Her interpolation method is compatible with the inclusion $x \mapsto d^{1/2p} x d^{1/2p}$, where $x \in \mathfrak{M}_\varphi$ (see [16] for further details).

Izumi's construction in [6] simultaneously generalizes the work of both Terp and Kosaki. It works for a general normal, semi-finite, faithful weight, and provides a family of embeddings depending on a complex parameter $\alpha \in \mathbb{C}$. For each $\alpha \in \mathbb{C}$, the intersection of M_* and M is

$$L_\alpha = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{there exists a unique functional } \varphi_x^\alpha \in M_* \text{ such that} \\ \varphi_x^\alpha(y^* z) = \big(x J \Delta^{\overline{\alpha}} \Lambda(y) \mid J \Delta^{-\alpha} \Lambda(z)\big) \\ \text{for all } y, z \in \pi_l(\mathfrak{A}_0) \end{array} \right\}.$$

This is a Banach space when considered with the norm

$$\|x\|_{L_\alpha} = \max\{\|x\|_\infty, \|\varphi_x^\alpha\|_1\},$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are the norms on M and M_*, respectively.

We recall the following results, which can be found as Propositions 2.3, 2.4 and 2.6 in [6].

Proposition 2.5. *For any $\alpha \in \mathbb{C}$, we have*

$$\pi_l(\mathfrak{A}_0^2) \subset L_\alpha$$

and for $y, z \in \pi_l(\mathfrak{A}_0)$, we have

$$\varphi_{y^* z}^\alpha = \omega_{J\Delta^{-\overline{\alpha}}\Lambda(y), J\Delta^\alpha \Lambda(z)}$$

where $\omega_{\xi,\eta}$ is the functional $(\cdot\, \xi \mid \eta)$.

Proposition 2.6. *The maps*

$$\iota_\alpha : L_\alpha \to M, x \mapsto x \text{ and } j_\alpha : L_\alpha \to M_*, x \mapsto \varphi_x^\alpha,$$

are norm-decreasing and injective. The set $i_\alpha(\pi_l(\mathfrak{A}_0^2))$ is σ-weakly dense in M and the set $j_\alpha(\pi_l(\mathfrak{A}_0^2))$ is norm dense in M_*.

Proposition 2.7. *For each $\alpha \in \mathbb{C}$, the Banach space L_α is a $(\pi_l(\mathfrak{A}_0), \pi_l(\mathfrak{A}_0))$-bimodule. If $a, b \in \pi_l(\mathfrak{A}_0)$ and $x \in L_\alpha$, then $axb \in L_\alpha$ and*

$$\varphi_{axb}^\alpha = \sigma_{-i\alpha - i/2}^\varphi(a)\, \varphi_x^\alpha\, \sigma_{-i\alpha + i/2}^\varphi(b),$$

where for $u, v \in M$, $\psi \in M_*$, the symbol $u\psi v$ means an element of M_* defined by the formula $\langle u\psi v, a\rangle = \psi(vau)$, for $a \in M$.

To obtain a compatible pair in the sense of interpolation theory, the following embeddings are used: $i_{-\alpha}^* : M_* \hookrightarrow L_{-\alpha}^*$ and $j_{-\alpha}^* : M \hookrightarrow L_{-\alpha}^*$, where $i_{-\alpha}^*$ is the restriction of the usual adjoint map of $i_{-\alpha}$ to M_*. Explicitly,

$$\langle y, i_{-\alpha}^*(\psi)\rangle_{L_{-\alpha}, L_{-\alpha}^*} = \psi(y), \quad y \in L_{-\alpha}, \psi \in M_*;$$
$$\langle y, j_{-\alpha}^*(x)\rangle_{L_{-\alpha}, L_{-\alpha}^*} = \varphi_y^{-\alpha}(x), \quad y \in L_{-\alpha}, x \in M.$$

These maps are norm-decreasing and injective. This compatible pair will be denoted $(M, M_*)_\alpha$ and is used to define non-commutative L_p-spaces

$$L_p(M, \varphi)_\alpha = C_{1/p}(M, M_*)_\alpha, \quad 1 < p < \infty, \alpha \in \mathbb{C},$$

using the complex interpolation method, which is an exact interpolation functor of exponent $\frac{1}{p}$. See [1] and [6] for more information about the complex interpolation method.

Remark 2.8. The set L_α defined above is the intersection of M and M_* in $L_{-\alpha}^*$. (Corollary 2.13, [6])

Remark 2.9. Only the case $\alpha = -1/2$ will be used here; for this reason subscript and superscript α's will often be omitted. In particular, φ_x shall stand for $\varphi_x^{-1/2}$ as defined above. Especially important in what follows are $L = L_{-1/2}$, the intersection of L_1 and L_∞, and $j^* = j^*_{1/2}$, the inclusion of M into $L^*_{1/2}$, the space containing all of the interpolation spaces that arise. The notation $L_p(M,\varphi)_\alpha$ indicates the L_p space constructed using interpolation, whereas $L_p(M,\varphi)$ denotes a spatial L_p space in the sense of Hilsum.

Izumi showed (as Theorem 3.8 in [6]) that the L_p spaces defined in this way are isometrically isomorphic to the interpolation spaces of Terp and thus to the L_p spaces of Haagerup and Hilsum.

Theorem 2.10. *Let M be a von Neumann algebra with normal, semi-finite, faithful weight φ. Let $\alpha, \beta \in \mathbb{C}$. There exists an isometric isomorphism*

$$U_{p,\beta,\alpha} : L_p(M,\varphi)_\alpha \to L_p(M,\varphi)_\beta, \quad 1 < p < \infty,$$

such that

$$U_{p,\beta,\alpha}(j^*_{-\alpha}(a)) = j^*_{-\beta}\left(\sigma^\varphi_{i\frac{r'-r}{p}-(s'-s)}(a)\right)$$

for any $a \in \pi_l(\mathfrak{A}_0^2)$, where $\alpha = r + is$, $\beta = r' + is'$, $r, r', s, s' \in \mathbb{R}$.

Remark 2.11. We will be interested in the following special case of Theorem 2.10. We have

$$U_{p,0,-1/2}(j^*(a)) = j^*_0(\sigma^\varphi_{\frac{i}{2p}}(a)).$$

Using Theorem 27 of [16], we can compose to get an isometric isomorphism $U_p : L_p(M,\varphi)_{-1/2} \to L_p(M,\varphi)$ such that for $a \in \pi_l(\mathfrak{A}_0^2)$, we have

$$U_p(j^*(a)) = d^{1/2p}\sigma^\varphi_{\frac{i}{2p}}(a)d^{1/2p}.$$

For $p \geq 2$, $a \in \pi_l(\mathfrak{A}_0^2)$, Theorem 2.4 allows this to be written as

$$U_p(j^*(a)) = ad^{1/p}.$$

Remark 2.12. Combining Remark 2.11 with Theorem 2.3 yields an isometry from $L_2(M,\varphi)_{-1/2}$ onto $\mathcal{H}$, which maps $U_2^{-1}(xd^{1/2})$ onto $\Lambda(x)$, for $x \in \pi_l(\mathfrak{A}_0^2)$.

3. The Hausdorff–Young Inequality

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \varphi_r)$ be a locally compact quantum group, and let $\hat{\mathbb{G}} = (L_\infty(\hat{\mathbb{G}}), \hat{\Gamma}, \hat{\varphi}, \hat{\varphi}_r)$ be its dual locally compact quantum group. Let ψ (respectively $\hat{\psi}$) be a normal semi-finite weight on the commutant $L_\infty(\mathbb{G})'$ (respectively $L_\infty(\hat{\mathbb{G}})'$). For example, one could take $\psi = \varphi(J \cdot J)$ (respectively $\hat{\psi} = \hat{\varphi}(\hat{J} \cdot \hat{J})$). Let $d = \frac{d\varphi}{d\psi}$ and $\hat{d} = \frac{d\hat{\varphi}}{d\hat{\psi}}$. Let $L_p(\mathbb{G})$ (respectively $L_q(\hat{\mathbb{G}})$) be the spatial non-commutative L_p space associated to the pair $(L_\infty(\mathbb{G}), \psi)$ (respectively $(L_\infty(\hat{\mathbb{G}}), \hat{\psi})$). As before, $L_p(\mathbb{G})_{-1/2}$ denotes an L_p space constructed by interpolation. Let $\mathfrak{A}_0$ be the Tomita algebra associated with the pair $(L_\infty(\mathbb{G}), \varphi)$ and let $\pi_l(\mathfrak{A}_0^2) = \operatorname{span}\{y^*z : y, z \in \pi_l(\mathfrak{A}_0)\}$.

The results from Section 2 describe the image of $x \in \pi_l(\mathfrak{A}_0^2)$ under the inclusions into $L_p(\mathbb{G})_{-1/2}$ and $L_p(\mathbb{G})$. To prove the Hausdorff–Young inequality, it is necessary to identify the intersection $L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2}$ and approximate its elements by elements in $\pi_l(\mathfrak{A}_0^2)$.

Theorem 3.1. *For all $x \in \mathfrak{N}_\varphi$, $2 \leq p < \infty$,*

$$U_p(j^*(x)) = xd^{1/p}.$$

Proof. First we consider the case $p = 2$. Let $x \in \mathfrak{N}_\varphi$. By Theorem 2.3 and Remark 2.11, we have that $xd^{1/2} \in L_2(\mathbb{G})$ and $U_2^{-1}(xd^{1/2}) \in L_2(\mathbb{G})_{-1/2}$. By Theorem 2.1, there exists a sequence $\{x_n\} \subset \pi_l(\mathfrak{A}_0^2)$ such that $x_n \to x$ strongly, $\|x_n\| \leq \|x\|$, and $\|\Lambda(x_n) - \Lambda(x)\|_2 \to 0$. By Remark 2.12, convergence in $\mathcal{H}$ implies convergence in $L_2(M,\varphi)_{-1/2}$. As the inclusion of $L_2(M,\varphi)_{-1/2}$ into $L^*_{1/2}$ is continuous, convergence in $L_2(\mathbb{G})_{-1/2}$ in turn implies that $U_2^{-1}(x_nd^{1/2}) \to U_2^{-1}(xd^{1/2})$ in the $\sigma(L^*_{1/2}, L_{1/2})$-topology.

The inclusion j^* of $L_\infty(\mathbb{G})$ into $L^*_{1/2}$ is the adjoint of the map $j : L_{1/2} \to L_\infty(\mathbb{G})_*$. Thus the map j^* is $\sigma(L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*) - \sigma(L^*_{1/2}, L_{1/2})$ continuous. As $\{x_n\}$ is a strongly convergent, $\|\cdot\|_\infty$-bounded sequence,

this sequence also converges in the $\sigma(L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*)$-topology. Thus $j^*(x_n) \to j^*(x)$ in the $\sigma(L^*_{1/2}, L_{1/2})$-topology.

As $x_n \in \mathfrak{A}_0^2$, applying Remark 2.11 yields that $j^*(x_n) = U_2^{-1}(x_n d^{1/2})$. Since the $\sigma(L^*_{1/2}, L_{1/2})$-topology is a Hausdorff topology, it follows that $j^*(x) = U_2^{-1}(xd^{1/2})$, i.e., $U_2(j^*(x)) = xd^{1/2}$ for $x \in \mathfrak{N}_\varphi$.

Now we consider the case $p > 2$. Let $x \in \mathfrak{N}_\varphi$ and again let $\{x_n\}$ be a sequence $\{x_n\} \subset \pi_l(\mathfrak{A}_0^2)$ such that $x_n \to x$ strongly, $\|x_n\| \le \|x\|$, and $\|\Lambda(x_n) - \Lambda(x)\|_2 \to 0$. By Theorem 2.3, we have that $xd^{1/p} \in L_p(\mathbb{G})$ and thus $(x_n - x)d^{1/p} \in L_p(\mathbb{G})$.

The three line theorem and Proposition 25 from [16] are now used to estimate $\|(x_n - x)d^{1/p}\|_p$. Let $a_n \in \mathbb{R}$ be such that $e^{a_n} = \|\Lambda(x_n - x)\|_2^{-2}\|x_n - x\|^2$. We now repeat the proof of Theorem 26 in [16] but with the function $F(z)$ replaced by $G_n(z) = e^{a_n(z-1/p)}F(z)$. This yields

$$\|(x_n - x)d^{1/p}\|_p \le \|\Lambda(x_n - x)\|_2^{2/p}\|x_n - x\|^{1-2/p}.$$

As $\|x_n - x\| \le 2$, the convergence of $x_n d^{1/2} \to xd^{1/2}$ in $\|\cdot\|_2$-norm implies that $x_n d^{1/p} \to xd^{1/p}$ in $\|\cdot\|_p$-norm. Proceeding as in the $p = 2$ case, we obtain the stated result. □

Remark 3.2. This theorem will be used in the following two ways. If $x \in \mathfrak{N}_\varphi \subset L_\infty(\mathbb{G})$, then the corresponding element of $L_2(\mathbb{G})$ is $xd^{1/2}$ and this inclusion agrees with the interpolation method being used. If $\omega \in \mathcal{I}$ and $q \ge 2$, then $\lambda(\omega)$ lies in $\mathfrak{N}_{\hat{\varphi}} \subset L_\infty(\hat{\mathbb{G}})$, the corresponding element of $L_q(\mathbb{G})$ is $\lambda(\omega)\hat{d}^{1/q}$ and this inclusion agrees with the interpolation method being used.

Lemma 3.3. *The intersection L is a subspace of $\mathfrak{N}_\varphi$.*

Proof. **Claim:** Given $x \in L$, there exists a net (x_k) in $L \cap \mathfrak{N}_\varphi$, such that

(1) $\sup_k \|x_k\| < \infty$
(2) $x_k \to x$ σ-strongly
(3) $\varphi_{x_k} \to \varphi_x$ in $\|\cdot\|_1$-norm

This will be shown in a manner similar to Lemma 9 in [16] and the proof of Theorem 2.5 in [6].

By the Kaplansky Density Theorem, there is a net $\{f_k\}_k$ of self-adjoint elements in the unit ball of $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ such that $f_k \to 1$ in the σ-strong*-topology. Define the net of elements $\{e_k\}_k$ by

$$e_k = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sigma_t^\varphi(f_k)\, \mathrm{d}t.$$

Proposition 2.16 in [12] shows that $\{e_k\}_k \subset \pi_l(\mathfrak{A}_0)$, and that for every $\alpha \in \mathbb{C}$

$$\sigma_\alpha^\varphi(e_k) \to 1 \text{ in the } \sigma\text{-strong}^* \text{ topology},$$
$$\|\sigma_\alpha^\varphi(e_k)\| \le e^{(Im\ \alpha)^2}.$$

Set $x_k = e_k x \sigma_{-i}^\varphi(e_k)$. By Proposition 2.7, L is an $\pi_l(\mathfrak{A}_0)$-bimodule. Thus $x_k \in L$ and $\varphi_{x_k} = e_k \varphi_x e_k$. As $\sigma_{-i}^\varphi(e_k) \in \pi_l(\mathfrak{A}_0) \subset \mathfrak{N}_\varphi$, it follows that $x_k \in L \cap \mathfrak{N}_\varphi$. As $\{e_k\}$ and $\{\sigma_{-i}^\varphi(e_k)\}$ are bounded nets converging strongly to 1, $\{x_k\}$ is a bounded net converging strongly to x.

As $L_\infty(\mathbb{G})$ is a von Neumann algebra in standard form, there exist $\xi, \eta \in \mathcal{H}$ such that $\varphi_x = (\cdot\ \xi \mid \eta)$. Fix $y \in L_\infty(\mathbb{G})$.

$$\begin{aligned}
|\varphi_{x_k}(y) - \varphi_x(y)| =& (e_k y e_k \xi \mid \eta) - (y\xi \mid \eta)| \\
=& |(y(e_k\xi) \mid \eta) - (y\xi \mid e_k\eta) + (y\xi \mid e_k\eta) - (y\xi \mid \eta)| \\
=& |(y(e_k\xi - \xi) \mid e_k\eta) + (y\xi \mid (e_k\eta - \eta))| \\
\le& \|y\|(\|e_k\xi - \xi\|\ \|\eta\| + \|\xi\|\ \|e_k\eta - \eta\|)
\end{aligned}$$

and thus $\|\varphi_{x_k} - \varphi_x\|_1 \to 0$. This completes the proof of the claim.

It remains to show that that $x \in \mathfrak{N}_\varphi$. As $x_k - x \in L$, the intersection of $L_\infty(\mathbb{G})$ and $L_\infty(\mathbb{G})_*$ in L^*, the norm $\|j^*(x_k - x)\|_2$ can be estimated using the norms $\|\varphi_{x_k} - \varphi_x\|_1$ and $\|x_k - x\|_\infty$. Consider the function $f_k(z) = e^{a_k(z^2-(1/2)^2)}j^*(x_k - x)$, where $e^{a_k} = \|x_k - x\|_1\|x_k - x\|_\infty^{-1}$. The definition of the norm on the interpolation space $L_p(\mathbb{G})_{-1/2}$ yields that

$$\|j^*(x_k - x)\|_2 \le \|j^*(x_k - x)\|_1^{3/4}\|j^*(x_k - x)\|_\infty^{1/4} = \|\varphi_{x_k} - \varphi_x\|_1^{3/4}\|x_k - x\|_\infty^{1/4}.$$

As $x_k - x$ is $\|\cdot\|_\infty$-bounded and $\varphi_{x_k} \to \varphi_x$ in $\|\cdot\|_1$, it follows that

$$\|j^*(x_k) - j^*(x)\|_2 \to 0.$$

Let $\xi \in \mathcal{H}$ be the image of $j^*(x)$ under the isometry from $L_2(\mathbb{G})_{-1/2}$ onto $\mathcal{H}$, as given by Remark 2.12. For any $\eta \in \mathfrak{A}'$,

$$\pi_r(\eta)\xi = \lim \pi_r(\eta)\Lambda(x_k) = \lim x_k\eta = x\eta.$$

Thus ξ is left bounded and $x = \pi_l(\xi)$. By Theorem 2.5 of [13], $x \in \mathfrak{N}_\varphi$ and $\xi = \Lambda(x)$. □

Lemma 3.4. *Let $\mathbb{G}$ be a locally compact quantum group and let $\mathcal{I}$ be as in Section 2.2. If $x \in L$, then $\varphi_x \in \mathcal{I}$ and*

$$\hat{\Lambda}(\lambda(\varphi_x)) = \xi(\varphi_x) = \Lambda(x).$$

Proof. Let $x \in L$ which implies by Lemma 3.3 that $x \in \mathfrak{N}_\varphi$. The definition of L provides a normal linear functional $\varphi_x \in L_\infty(\mathbb{G})_*$ such that for all $y, z \in \pi_l(\mathfrak{A}_0^2)$

$$\begin{aligned}
\varphi_x(y^*z) &= \left(xJ\Delta^{-1/2}\Lambda(y)\middle|J\Delta^{1/2}\Lambda(z)\right) \\
&= (x\Lambda(\sigma_{-i}(y^*))|\Lambda(z^*)) \\
&= \varphi\left(zx\sigma_{-i}(y^*)\right)) \\
&= \varphi\left(y^*zx\right) \\
&= (\Lambda(x)|\Lambda((y^*z)^*))
\end{aligned}$$

Since $zx \in \mathfrak{M}_\varphi$, Proposition 2.2 justifies the second last equality.

The above is now used to show that $\varphi_x(a^*) = (\Lambda(x)|\Lambda(a))$ for all $a \in \mathfrak{N}_\varphi$. By Theorem 2.1, there exists a sequence $\{a_n\} \subset \pi_l(\mathfrak{A}_0^2)$ such that $\|\Lambda(a_n) - \Lambda(a)\|_2 \to 0$ and $a_n^* \to a^*$ σ-weakly. As φ_x is a normal functional and $a_n^* \to a^*$ σ-weakly, $\varphi_x(a_n^*) \to \varphi_x(a^*)$, and then

$$\varphi_x(a^*) = \lim \varphi_x(a_n^*) = \lim (\Lambda(x)|\Lambda(a_n)) = (\Lambda(x)|\Lambda(a)).$$

The result now follows from the construction of the map $\hat{\Lambda}$ (see Section 2.2). □

The L_p-Fourier transform can now be defined and shown to satisfy the Hausdorff–Young inequality.

Theorem 3.5. *Let $1 \le p \le 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. The L_p-Fourier transform of $\mathbb{G}$ is the extension of the map*

$$\mathcal{F}_p\left(U_p\left(j^*(a)\right)\right) = \lambda\left(\varphi_a\right)\hat{d}^{1/q}, \qquad a \in L,$$

to a contraction $\mathcal{F}_p : L_q(\mathbb{G}, \hat{\varphi}) \to L_p(\hat{\mathbb{G}}, \hat{\varphi})$ and thus

$$\|\mathcal{F}_p\left(U_p\left(j^*(a)\right)\right)\|_q \le \|ad^{1/p}\|_p.$$

Remark 3.6. For $a \in \pi_l(\mathfrak{A}_0^2)$, there is a more explicit description of $\mathcal{F}_p$:

$$\mathcal{F}_p\left(d^{1/2p}\sigma^\varphi_{i/2p}(a)d^{1/2p}\right) = \lambda(\varphi_a)\hat{d}^{1/q}, \quad a \in \pi_l(\mathfrak{A}_0^2).$$

If φ is a state, we have the simpler expression

$$\mathcal{F}_p(ad^{1/p}) = \lambda(\varphi_a)\hat{d}^{1/q}, \qquad a \in L_\infty(\mathbb{G}).$$

Proof of Theorem 3.5. The case $p = 1$ is obvious. For $\omega \in L_\infty(\mathbb{G})_*$, $\mathcal{F}_1$ is the map $\omega \mapsto \lambda(\omega) = (\omega \otimes \iota)\,W$, which is a contraction.

We now consider the case $p = 2$. By Theorem 2.3, we can identify the Hilbert spaces $\mathcal{H}_\varphi$ and $L_2(\mathbb{G})$ by using the isometry $\alpha : \Lambda(x) \mapsto xd^{1/2}$, for $x \in \mathfrak{N}_\varphi$. Similarly, we can identify $L_2(\hat{\mathbb{G}})$ and $\mathcal{H}_{\hat{\varphi}}$ by using the isometry $\beta : \lambda(\omega) \mapsto \hat{\Lambda}(\lambda(\omega))$, for $\lambda(\omega) \in \mathfrak{N}_{\hat{\varphi}}$. Furthermore, from Lemma 3.4 and the construction of the dual Haar weight, the map $\beta \circ \mathcal{F}_2 \circ \alpha : \mathcal{H}_\varphi \to \mathcal{H}_{\hat{\varphi}}$ is the identity map. This implies that $\mathcal{F}_2$ is an isometry from $L_2(\mathbb{G})$ onto $L_2(\hat{\mathbb{G}})$.

The next step is to show that $\mathcal{F}_1$ and $\mathcal{F}_2$ agree on $L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2} \subset L^*_{1/2}$.

Suppose that $x \in L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2}$, with $U_1(x) = \psi \in L_\infty(\mathbb{G})_*$ and $U_2(x) = \alpha(\xi) \in L_2(\mathbb{G})$ for $\xi \in \mathcal{H}_\varphi$. Let z_i be a sequence in $\mathfrak{A}_0^2$ such that $\|\Lambda(z_i) - \xi\|_2 \to 0$. As the inclusions into $L^*_{1/2}$ are continuous, this implies that $j^*(z_i) \to x$ in the $\sigma(L^*_{1/2}, L_{1/2})$-topology. Therefore for $y \in \pi_l(\mathfrak{A}_0^2)$,

$$\langle y, \psi\rangle_{L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*} = (\xi \mid \Lambda(y^*))$$

since

$$\begin{aligned} \langle y, \psi\rangle_{L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*} &= \langle y, x\rangle_{L_{1/2}, L^*_{1/2}} \\ = \lim\langle y, j^*(z_i)\rangle_{L_{1/2}, L^*_{1/2}} &= \lim\langle y, \varphi_{z_i}\rangle \\ = \lim(\Lambda(z_i) \mid \Lambda(y^*)) &= (\xi \mid \Lambda(y^*)). \end{aligned}$$

Suppose that $y \in \mathfrak{N}^*_\varphi$. By Theorem 2.1, there exists a sequence $\{y_n^*\}$ in $\pi_l(\mathfrak{A}_0^2)$ such that $\|\Lambda(y_n^*) - \Lambda(y^*)\|_2 \to 0$ and $y_n \to y$ σ-weakly. Thus $\psi(y_n) \to \psi(y)$ and it follows that

$$\langle y, \psi\rangle = \lim\langle y_n, \psi\rangle = \lim(\xi \mid \Lambda(y_n^*)) = (\xi \mid \Lambda(y^*)), \qquad \forall y \in \mathfrak{N}^*_\varphi.$$

Therefore $\psi \in \mathcal{I}$ and $\xi(\psi) = \xi$. However by the construction of the Haar weight on $L_\infty(\hat{\mathbb{G}})$, this implies that $\lambda(\psi) \in \mathfrak{N}_{\hat{\varphi}}$ and, by Theorem 2.3, $\lambda(\psi)\hat{d}^{1/2} \in L_2(\hat{\mathbb{G}})$. Then $\beta(\lambda(\psi)\hat{d}^{1/2}) = \hat{\Lambda}(\lambda(\psi)) = \xi(\psi) = \xi$. As $\beta \circ \mathcal{F}_2 \circ \alpha = id_{\mathcal{H}_\varphi}$, this implies $\beta^{-1}(\xi) = \mathcal{F}_2(\alpha(\xi))$ or $\mathcal{F}_2(U_2(x)) = \lambda(\psi)\hat{d}^{1/2}$.

Thus $\mathcal{F}_1(U_1(x)) = \lambda(\psi)$ and $\mathcal{F}_2(U_2(x)) = \lambda(\psi)\hat{d}^{1/2}$. By Remark 3.2, these two elements are the same when considered as elements of $\hat{L}^*_{1/2}$, and the Fourier transform is well-defined as a map

$$L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2} \to L_2(\hat{\mathbb{G}})_{-1/2} \cap L_\infty(\hat{\mathbb{G}})_{-1/2}.$$

The Hausdorff–Young inequality now follows after interpolating between the cases $p = 1$ and $p = 2$ as the complex interpolation method is an exact interpolation functor. □

This work is part of my doctoral dissertation at the University of Illinois at Urbana-Champaign. I would like to thank my advisor Professor Z.-J. Ruan for many helpful discussions.

References

[1] J. Bergh and J. Löfström, Interpolation spaces: An Introduction, Springer-Verlag, 1976.

[2] A. Connes, *On the spatial theory of von Neumann algebras*, J. Func. Anal., **35** (1980), 153–164.

[3] G. Folland, A First Course in Abstract Harmonic Analysis, Studies in Advanced Mathematics, CRC Press, 1995.

[4] U. Haagerup, *A density theorem for left Hilbert algebras*, Algèbres d'opérateurs, 170–179, Lecture Notes in Math., **725**, Springer, Berlin, 1979.

[5] M. Hilsum, *Les espaces L_p d'une algèbre de von Neumann définies par la dérivée spatiale*, J. Funct. Anal., **40** (1981), 151–169.

[6] H. Izumi, *Constructions of non-commutative L^p-spaces with a complex parameter arising from modular actions*, Internat. J. Math., **8**, No. 8 (1997) 1029–1066.

[7] S. Goldstein and J.M. Lindsay, *Markov semigroups KMS-symmetric for a weight*, Math. Ann. **313** (1999), 39–67.

[8] B. Kahng, *Fourier Transform on Locally Compact Quantum Groups*, 2007, preprint, `arXiv:0708.3055v1 [math.OA]`.

[9] H. Kosaki, *Applications of the Complex Interpolation Method to a von Neumann Algebra: Non-commutative L^p-spaces*, J. Funct. Anal., **56** (1984), 29–78.

[10] R. Kunze, *L_p Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc., **89** (1958), 519–540.

[11] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand., **92** (2003), 68–92.

[12] S. Stratila, Modular Theory in Operator Algebras, Abacus Press, Tunbridge Wells, 1981.

[13] M. Takesaki, Theory of Operator Algebras II, Encyclopaedia of Mathematical Sciences, **125**, Springer-Verlag, 2003.

[14] M. Terp, *L^p spaces associated with von Neumann algebras* (Notes), Report No. 3a+3b, Kobenhavns Univ. Matematiske Institut, June 1981.

[15] M. Terp, *L^p Fourier transformation on non-unimodular locally compact groups*, Kobenhavns Univ. Matematiske Institut, preprint, 1980.

[16] M. Terp, *Interpolation between a von Neumann algebra and its predual*, J. Operator Theory, **8** (1982), 327–360.

[17] A. Van Daele, *The Fourier transform in quantum group theory*, 2007, preprint, `arXiv:0609502V3 [math.RA]`